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Far-field description of the flow produced by a source of both momentum and mass

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The far-field description of the flow produced by a source of both momentum and mass is given in the form of a coordinate expansion. The first term of the expansion corresponds to the classical self-similar solution first written by Landau. The next two terms, associated with the non-zero rate of mass, are found in the paper.

1. Introduction

The existence of self-similar solution of the Navier–Stokes equations with the velocity decreasing as $1/r$ was first shown by Slezkin (1934). Landau (1944) revealed that this solution describes the steady flow induced by a point source of momentum placed in an unbounded fluid at rest; see Landau & Lifshitz (1987, § 23), or Sherman (1990, § 9.4), for details. Seven years after Landau, Squire (1951) published the same solution, together with the solution of the corresponding problem when the source is also a point source of heat. Thereafter, this solution is referred to frequently as the Landau–Squire one (LS below) in the fluid mechanics literature.

The LS-solution has been developed in the literature. The analysis of the transient cases, for which the point source of momentum is turned on suddenly at time zero, was carried out numerically by Sozou & Pickering (1977). Sozou (1979) found an analytic solution for such a creeping unsteady jet. Unsteady particle trajectories were examined by Cantwell (1981). Pickering & Sozou (1979) investigated the steady laminar flow produced by a point source of momentum in a spherical envelope. Recently, the steady flow due to a point force arose in the study of thermocapillary convection produced by a stationary bubble in a linear temperature field, see Balasubramaniam & Subramanian (2004).

If the size of the source is finite, the classical LS-result becomes the first term of an expansion in powers of the ratio of this size to the distance from the source. It is well known that the total mass flux through a closed surface surrounding the origin calculated from the LS-solution is zero. An attempt to incorporate the non-zero mass rate in the far-field description was first made by Rumer (1953), who used for this purpose the next term in the asymptotic coordinate expansion (as cited in Landau & Lifshitz 1987, § 23, p. 83). A revision of the solution obtained by Rumer reveals unremovable logarithmic singularities of the velocity field at the axis, and made it necessary to reconsider this problem.

2. Formulation

Consider a steady flow induced by a source of both momentum, of intensity $J\boldsymbol{e}_x$, and mass, of intensity M, placed in an unbounded environment of the same fluid with the density ρ and the kinematic viscosity ν. It is assumed in this study that there are no losses of momentum. One might imagine a tiny tube emerging from the fluid with slip velocity conditions at its outer surface or, for example, a porous spherical ball, fixed in space, with a non-uniform surface velocity distribution to provide a non-zero momentum flux in some direction. This paper deals with the far-field description of such flows and the near-field details do not affect the form of the far-field solution in first approximations†, as we will show below.

The resultant flow field is assumed to be axisymmetric and a system of spherical coordinates with the radial coordinate r' measured from the source and the angle θ measured from the direction of the momentum $\boldsymbol{e}_x$ is used to describe the flow, $0 \leqslant \theta \leqslant \pi$. To account for the source we require two conditions: that the conservation of momentum,

$$J = 2\pi r'^2 \int_0^{\pi} \left\{ \rho V_r (V_r \cos\theta - V_\theta \sin\theta) + p\cos\theta - \tau_{rr}\cos\theta + \tau_{r\theta}\sin\theta \right\} \sin\theta \, \mathrm{d}\theta, \quad (2.1)$$

and the conservation of mass,

$$M = 2\pi r'^2 \int_0^{\pi} \rho V_r \sin\theta \, \mathrm{d}\theta, \quad (2.2)$$

are satisfied at large r'. Here τ_{rr} and $\tau_{r\theta}$ are the components of the stress tensor.

Let us define the characteristic length and velocity using the relations $M = 2\pi\rho U_c l_c^2$ and $U_c = \nu/l_c$, namely

$$l_c = \frac{M}{2\pi\rho\nu}, \qquad U_c = \frac{2\pi\rho\nu^2}{M}, \quad (2.3)$$

and measure the pressure field using ρU_c^2. In the above mentioned case of tube emerging the jet, $l_c = aRe/2$, where Re is the Reynolds number of the tube flow based on the average velocity and the tube radius a.

Using l_c and U_c as scales for the spatial coordinates and for the velocity, $K = J/2\pi\rho\nu^2$ remains the only parameter entering in the far-field description. Introducing the non-dimensional stream function ψ (measured using $\psi_c = M/2\pi\rho$), the governing Navier–Stokes equation written in terms of $r = r'/l_c$ and $\zeta = \cos\theta$ takes the form

$$\frac{1}{r^2}\frac{\partial(\psi, \mathrm{D}^2\psi)}{\partial(r,\zeta)} + \frac{2\mathrm{D}^2\psi}{r^2}\left(\frac{\zeta}{1-\zeta^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\zeta}\right) = \mathrm{D}^4\psi, \quad (2.4)$$

where

$$\mathrm{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial\zeta^2}.$$

Here the stream function is defined by

$$V_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\zeta}, \quad V_\theta = -\frac{1}{r\sqrt{1-\zeta^2}}\frac{\partial\psi}{\partial r}.$$

The value $-\mathrm{D}^2\psi/r\sqrt{1-\zeta^2}$ is the azimuthal component of the vorticity.

† More precisely, in approximations considered the near-field details affect only the value of C_1 appearing in (3.12).

Equation (2.4) needs to be supplemented by the following boundary conditions at the axis:

$$\psi|_{\zeta=-1} = 1, \quad \psi|_{\zeta=1} = 0, \quad \lim_{\zeta\to\pm1} \sqrt{1-\zeta^2}\,\frac{\partial^2\psi}{\partial\zeta^2} = 0. \tag{2.5}$$

The first and the second conditions represent the conservation of mass (2.2). The last condition in (2.5) can be substituted by the requirement that radial velocity, proportional to $\partial\psi/\partial\zeta$, is finite at the axis, $\zeta=\pm1$.

3. Asymptotic solution for $r \gg 1$

The LS-solution describing the point source of momentum with zero mass rate, $M=0$, is the exact solution of (2.4)–(2.5) valid for all r. In terms of the stream function it takes the form

$$\psi = r f_0(\zeta), \quad \text{with} \quad f_0 = 2(1-\zeta^2)/(A-\zeta), \tag{3.1}$$

where constant A is related to $K = J/2\pi\rho\nu^2$ by the relation

$$K = 8A\left(1 + \frac{4}{3(A^2-1)} - \frac{1}{2}A\ln\frac{A+1}{A-1}\right), \quad 1 < A < \infty, \tag{3.2}$$

see Landau & Lifshitz (1987). In what follows the constant A being used, instead of K, to characterize the flow field.

We should remark here that in spite of M being used to define the length scale l_c, the dimensional stream function ψ' describing the point source of momentum with zero mass rate remains independent of M,

$$\psi' = \frac{M}{2\pi\rho}\left(\frac{r'}{l_c}\right) f_0(\zeta) = \nu r' f_0(\zeta), \tag{3.3}$$

as it should be, where r' is the dimensional radial coordinate.

Consider the far flow field produced by the momentum source with non-zero M. To describe the asymptotic solution at large r the following coordinate expansion is suggested at $r \gg 1$:

$$\psi = r f_0(\zeta) + \ln(r) f_1(\zeta) + f_2(\zeta) + \cdots, \tag{3.4}$$

where the first term is given by the LS-solution (3.1). Notice that in the expansion proposed in Rumer (1953) the term proportional to $\ln r$ was not included. Substituting (3.4) into (2.4) and equating the same order terms of the orders of $r^{-4}\ln r$ and r^{-4} provides equations (A 1) and (A 2), respectively, given in the Appendix. Using the LS-solution (3.1) the homogeneous equation (A 1) written in terms of $h_1 = f_1^I$ becomes

$$a_1 h_1 + a_2 h_1^I + a_3 h_1^{II} + a_4 h_1^{III} = 0, \tag{3.5}$$

where the coefficients a_i are also given in the Appendix. Equation (3.5) is to be solved with the only condition that h_1 is finite at the axis, $\zeta=\pm1$.

It was found that equation (3.5) has the unique general solution regular at the axis which, aside from a constant factor, is of the form

$$\mathscr{H}_1 = \frac{A^2 - 2 + 3\zeta A - 3\zeta^2 A^2 + \zeta^3 A}{A(\zeta - A)^3}. \tag{3.6}$$

The other two general solutions $\mathscr{H}_2$ and $\mathscr{H}_3$, given in the Appendix, both have logarithmic singularities at the axis. Then, h_1 is given by

$$h_1 = C\mathscr{H}_1, \tag{3.7}$$

where the constant C remains undetermined in the frame of this approximation. It is easily seen that

$$\int_{-1}^{1} \mathscr{H}_1 \, \mathrm{d}\zeta = 0 \tag{3.8}$$

and the second term in (3.4) does not contain the mass, as it should due to its logarithmic with r behaviour. In terms of the stream function the solution is

$$f_1 = C\frac{(1-\zeta^2)(1-A\zeta)}{A(\zeta - A)^2}. \tag{3.9}$$

The calculation of constant C requires consideration of the next order term, f_2, in (3.4).

Using (3.1) and (3.9) in (A 2) gives

$$Ca_0 + a_1 h_2 + a_2 h_2^I + a_3 h_2^{II} + a_4 h_2^{III} = 0, \tag{3.10}$$

where $h_2 = f_2^I$ and a_0 is given in the Appendix. The third term in the expansion (3.4) contains the mass and, then, the condition

$$\int_{-1}^{1} h_2 \, \mathrm{d}\zeta = -1 \tag{3.11}$$

needs to be imposed.

The homogeneous part of (3.10) coincides with (3.5). It was found that the particular solution of (3.10), $\mathscr{H}_p$, together with $\mathscr{H}_2$ and $\mathscr{H}_3$, has logarithmic singularities at $\zeta = \pm 1$. Then, the general solution of (3.10) can be written in the form

$$h_2 = C_1\mathscr{H}_1 + h_*, \tag{3.12}$$

where

$$h_* = C_2\mathscr{H}_2 + C_3\mathscr{H}_3 + C\mathscr{H}_p \tag{3.13}$$

denotes the part of h_2 that is singular at the axis.

One can now use that the physically acceptable solution for h_2 must be regular at the axis. The main reason to include the term proportional to $\ln r$ in the coordinate expansion (3.4) is to employ the constant C to eliminate the singularity in (3.13). The following designations are applied below:

$$S_j^+ = \lim_{\zeta \to 1} \{\mathscr{H}_j / \ln(1-\zeta)\}, \quad S_j^- = \lim_{\zeta \to -1} \{\mathscr{H}_j / \ln(1+\zeta)\}, \quad \bar{S}_j = \int_{-1}^{1} \mathscr{H}_j \, \mathrm{d}\zeta, \tag{3.14}$$

where $j = 2$, 3 and p. Then, the conditions to remove the axis singularities together with the mass requirement (3.11) are

$$\left.\begin{aligned} C_2 S_2^+ + C_3 S_3^+ + C S_p^+ &= 0, \\ C_2 S_2^- + C_3 S_3^- + C S_p^- &= 0, \\ C_2 \bar{S}_2 + C_3 \bar{S}_3 + C \bar{S}_p &= -1. \end{aligned}\right\} \tag{3.15}$$

These conditions determine the value of C, together with C_2 and C_3, while the constant C_1 in (3.12) remains undetermined. Its value depends on the details of the flow near the source and can be determined only by matching the asymptotic solution at large r with the numerical solution at distances $r \sim O(1)$.

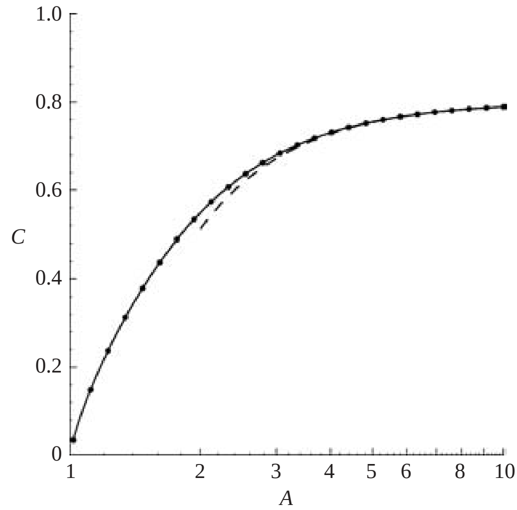

FIGURE 1. Computed values of C (solid line) and its asymptotic expansion (3.16) for large A (dots); dashed line, first two terms of (3.16).

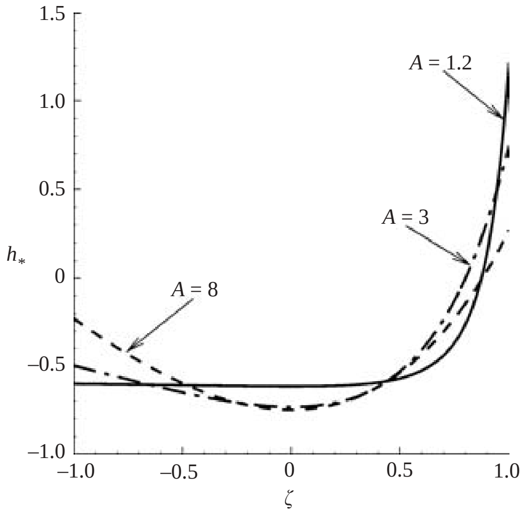

FIGURE 2. Profiles of h_* calculated for various A.

The partial solution $\mathscr{H}_p$ has a very large analytical form and a numerical treatment was applied in this study. The conditions

$$\mathscr{H}_p(0) = \mathscr{H}_p^I(0) = \mathscr{H}_p^{II}(0) = 0$$

were used to calculate numerically the particular solution $\mathscr{H}_p$. Clearly that the final result is independent of this choice. Shown in figure 1 with a solid line are the resultant values of C plotted as a function of A. Figure 2 shows h_* calculated for different A, where the distributions were normalized by the condition $h_*^I(0) = 0$. This condition can always be satisfied by adding to h_*, calculated numerically, some fraction of $\mathscr{H}_1$, because $\mathscr{H}_1^I(0) \neq 0$ for $A > 1$.

For $A \gg 1$ the value C is given by the expansion

$$C=\frac{4}{5}-\frac{604}{525}A^{-2}+\frac{38468}{55125}A^{-4}-0.48750A^{-6}+0.30037A^{-8}-0.20946A^{-10}$$
$$+\,0.12789A^{-12}-0.090725A^{-14}+0.053952A^{-16}-0.039681A^{-18}+\cdots. \quad (3.16)$$

This expansion, calculated analytically and represented in figure 1 with dots, describes with a high accuracy the numerical results for values of $A > 1.02$. The first two terms of expansion (3.16) are plotted in figure 1 with a dashed line. All coefficients in (3.16) were found in the exact form (as rationales). Nevertheless, the higher-order coefficients are presented here as approximate decimals for the sake of brevity. First terms of the h_*-expansion corresponding to (3.16) are given in the Appendix.

4. Constraint on the momentum flux

The far-field coordinate expansion of the stream function (3.4) leads to the following expansion for the total non-dimensional momentum flux:

$$\int_{-1}^{1} j_0\,\mathrm{d}\zeta+\frac{\ln r}{r}\int_{-1}^{1} j_1\,\mathrm{d}\zeta+\frac{1}{r}\int_{-1}^{1} j_2\,\mathrm{d}\zeta+\cdots, \quad (4.1)$$

where j_0, j_1 and j_2 are given in the Appendix; j_0 is the leading term in the local momentum flux while j_1 and j_2 represent the first and second corrections to this value. It is easily seen that the first integral $\int_{-1}^{1} j_0\,\mathrm{d}\zeta$ calculated using (3.1) coincides with the LS-value (3.2). The goal of this section is to demonstrate that the second and third terms appearing in (4.1) are both zero, or, equivalently, that the solution obtained in the previous section really satisfies the constraint on the momentum flux.

The constraint on j_1 is easily shown by using (3.1) and (3.6) in

$$j_1=C\big(2\zeta f_0^I-f_0-4\zeta\big)\mathscr{H}_1$$

and, as we obtained, $\int_{-1}^{1} j_1\,\mathrm{d}\zeta\equiv 0$.

Integrating equation (A 2) from -1 to ζ gives

$$(\zeta^2-1)h_2^{II}+2\zeta h_2^I-6h_2+(f_0h_2)^I+2f_0^Ih_2+\int_{-1}^{\zeta} S\,\mathrm{d}\zeta$$
$$=-2h_2^I(-1)+\big(3f_0^I(-1)-6\big)h_2(-1), \quad (4.2)$$

where $h_2=f_2^I$. Multiplying (4.2) by ζ and integrating again over ζ from -1 to 1 provides the following integral relation:

$$\int_{-1}^{1}(2\zeta f_0^I-f_0-4\zeta)h_2\,\mathrm{d}\zeta=\int_{-1}^{1}\frac{1}{2}(\zeta^2-1)S\,\mathrm{d}\zeta. \quad (4.3)$$

Analytical solutions (3.1) and (3.9) applied in expression (A 9) for j_2 reduce the third integral in (4.1) to the form

$$\int_{-1}^{1} j_2\,\mathrm{d}\zeta=C\left\{\frac{18A^2-4}{A}+(5-9A^2)\ln\left(\frac{A+1}{A-1}\right)\right\}+\int_{-1}^{1}\big(2\zeta f_0^I-f_0-4\zeta\big)h_2\,\mathrm{d}\zeta \quad (4.4)$$

and, as we obtained using (4.3) in order to eliminate h_2, $\int_{-1}^{1} j_2\,\mathrm{d}\zeta\equiv 0$.

5. Conclusions

A far-field description of the flow produced by the source of momentum with non-zero mass flow rate has been found in terms of the coordinate expansion (3.4). The resulting stream function rewritten in the dimensional form,

$$\psi' = \nu r' \frac{2(1-\zeta^2)}{A-\zeta} + C\frac{M}{2\pi\rho}\ln\left(\frac{r'}{l_c}\right)\frac{(1-\zeta^2)(1-A\zeta)}{A(\zeta-A)^2} + \frac{M}{2\pi\rho}f_2(\zeta) + \cdots, \tag{5.1}$$

is valid at distances $r' \gg l_c = M/2\pi\rho\nu$. The leading term in (5.1), found by both Landau and Squire, has the zero mass rate. The second term needs to be logarithmic in r in order to eliminate the singularities at the axis appearing in the third term associated with the non-zero mass flow rate. The third term includes the eigenfunction part multiplied by a factor dependent on the details of the flow at distances $r' \sim l_c$.

In the coordinate expansion proposed in Rumer (1953) the logarithmic term was not included, or, equivalently, the value $C = 0$ was applied. The analysis carried out in the present paper has shown that this choice leads to the velocity field with unremovable singularities at the axis in the term responsible for the non-zero mass rate. Even though the resultant singular velocity field is integrable to ensure a finite mass rate, such singular behaviour at the axis is unacceptable from the physical point of view.

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Appendix

Equations determining f_1 and f_2 in (3.4) are

$$(\zeta^2-1)f_1^{IV} + 4\zeta f_1^{III} - 4f_1^{II} + f_0 f_1^{III} + 4f_0^I f_1^{II} + 3f_0^{II} f_1^I = 0, \tag{A 1}$$

$$(\zeta^2-1)f_2^{IV} + 4\zeta f_2^{III} - 4f_2^{II} + f_0 f_2^{III} + 4f_0^I f_2^{II} + 3f_0^{II} f_2^I + S = 0, \tag{A 2}$$

where

$$S = \frac{1}{\zeta^2-1}\left[(\zeta^2-1)f_0^{III} + 4f_0^I - \frac{2\zeta}{\zeta^2-1}f_0 - 6\right]f_1 + \frac{f_0}{\zeta^2-1}f_1^I + \left(6 - f_0^I\right)f_1^{II}. \tag{A 3}$$

The coefficients appearing in (3.5) and (3.10) are

$$\left.\begin{aligned} a_0 &= 2\frac{12A - 9A^2\zeta + 3A^2\zeta^3 - \zeta^3 - 14A^3 - 6A^3\zeta^2 + 15A^4\zeta}{A(A-\zeta)^5}, \\ a_1 &= 12(1-A^2)/(A-\zeta)^3, \\ a_2 &= 4(\zeta^2 - 2\zeta A + 2 - A^2)/(A-\zeta)^2, \\ a_3 &= 2(-3\zeta^2 + 2\zeta A + 1)/(A-\zeta), \\ a_4 &= \zeta^2 - 1. \end{aligned}\right\} \tag{A 4}$$

Two singular general solutions of (3.5) are

$$\mathscr{H}_2 = \frac{-8 + 20A^2 - 18A^3\zeta + 6\zeta^2A^2}{(\zeta-A)^3} + \frac{-3A^3 + 6A - 9\zeta A^2 + 9\zeta^2A^3 - 3\zeta^3A^2}{(\zeta-A)^3}\ln\left(\frac{1+\zeta}{1-\zeta}\right), \tag{A 5}$$

$$\mathscr{H}_3 = \frac{A^2 - 2 + 3\zeta A - 3\zeta^2 A^2 + \zeta^3 A}{(A-\zeta)^3}$$
$$\times \int_0^{\zeta} \frac{(A-\zeta)^4(\zeta^3 A - 3\zeta^2 A^2 + 9\zeta A - 6A^3\zeta + (-6A^4 - 6 + 12A^2)\ln(A-\zeta))}{(A^2 - 2 + 3\zeta A - 3\zeta^2 A^2 + \zeta^3 A)^2 (1-\zeta^2)} \, d\zeta. \quad (\text{A }6)$$

The j_0, j_1 and j_2 appearing in (4.1) are

$$j_0 = \frac{1}{2}(1-\zeta^2) f_0^{II} - 2\zeta f_0^I + 2 f_0 + \zeta f_0^{I\,2} - \frac{3}{2} f_0 f_0^I + \frac{\zeta}{2(\zeta^2 - 1)} f_0^2, \quad (\text{A }7)$$

$$j_1 = \left(2\zeta f_0^I - f_0 - 4\zeta\right) f_1^I, \quad (\text{A }8)$$

$$j_2 = \left(2\zeta f_0^I - f_0 - 4\zeta\right) f_2^I + \left[-2 f_0^I - \frac{\zeta}{1-\zeta^2} f_0 + 4\right] f_1 + \left(2\zeta - \tfrac{1}{2} f_0\right) f_1^I + \tfrac{1}{2}(1-\zeta^2) f_1^{II}. \quad (\text{A }9)$$

With the help of the MAPLE program twenty terms of the h_*-expansion corresponding to (3.16) were calculated. Notice that C expands in powers of A^{-2} whereas h_*, given below, in powers of A^{-1}. The higher-order terms of the h_*-expansion take very large forms and the first seven terms are

$$h_* = \frac{3}{4}(\zeta^2 - 1) + \frac{1}{A} 2\zeta^3 + \frac{1}{A^2}\left(\frac{15}{4}\zeta^4 - \frac{27}{10}\zeta^2 + \frac{3}{20}\right) + \frac{1}{A^3}\left(6\zeta^5 - \frac{22}{3}\zeta^3\right)$$
$$+ \frac{1}{A^4}\left(\frac{35}{4}\zeta^6 - \frac{489}{35}\zeta^4 + \frac{17699}{4200}\zeta^2 + \frac{1759}{12600}\right) + \frac{1}{A^5}\left(12\zeta^7 - \frac{9503}{420}\zeta^5 + \frac{11969}{1050}\zeta^3\right)$$
$$+ \frac{1}{A^6}\left(\frac{63}{4}\zeta^8 - \frac{7193}{216}\zeta^6 + \frac{955783}{44100}\zeta^4 - \frac{802351}{220500}\zeta^2 - \frac{16819}{147000}\right) + \cdots. \quad (\text{A }10)$$

The first term in (3.16), $C = 4/5$, corresponds to the first three terms of (A 10).

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